15_11694 林明慧

(n≥2)

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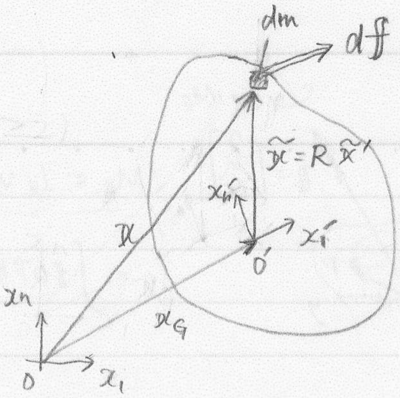
2016 1. 9

n次元剛体 並進・回転運動



$$\begin{cases} dm\,\ddot{x} = df \\ x = x_G + \tilde{x} = x_G + R\tilde{x}' \\ x_G := \dfrac{\int_V x\,dm}{\int_V dm} \end{cases}$$

$$df = df^{(i)} + df^{(e)}, \quad \int_V df^{(i)} = 0$$

$$R\int_V \tilde{x}'\,dm = \int_V \tilde{x}\,dm = \int_V x\,dm - x_G\int_V dm = 0$$

$$dm\,\ddot{x} = df \iff dm(\ddot{x}_G + \ddot{\tilde{x}}) = df$$

$$\Rightarrow \ddot{x}_G\int_V dm + \int_V \ddot{\tilde{x}}\,dm = \int_V df^{(i)} + \int_V df^{(e)}$$

$$\iff \boxed{\ddot{x}_G\int_V dm = \int_V df^{(e)}} \left(= \int_V \ddot{x}\,dm\right)$$

$$dm\,\ddot{x} = df \iff \begin{bmatrix} dm\,\ddot{x}_i \\ dm\,\ddot{x}_j \end{bmatrix} = \begin{bmatrix} df_i \\ df_j \end{bmatrix} \quad \text{for } \forall i, \forall j \in \{1 \cdots n\}$$

$$\iff \begin{bmatrix} x_i & dm\,\ddot{x}_i \\ x_j & dm\,\ddot{x}_j \end{bmatrix} = \begin{bmatrix} x_i & df_i \\ x_j & df_j \end{bmatrix}$$

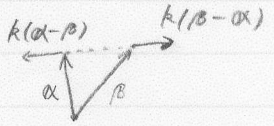
$$\Rightarrow \det\begin{bmatrix} x_i & dm\,\ddot{x}_i \\ x_j & dm\,\ddot{x}_j \end{bmatrix} = \det\begin{bmatrix} x_i & df_i \\ x_j & df_j \end{bmatrix}$$

$$\iff dm(x_i\ddot{x}_j - x_j\ddot{x}_i) = x_i\,df_j - x_j\,df_i$$

$$\iff dm\frac{d}{dt}(x_i\dot{x}_j - x_j\dot{x}_i) = \left(x_i\,df_j^{(i)} - x_j\,df_i^{(i)}\right) + \left(x_i\,df_j^{(e)} - x_j\,df_i^{(e)}\right)$$

$$\Rightarrow \int_V \frac{d}{dt}(x_i\dot{x}_j - x_j\dot{x}_i)\,dm = \int_V x_i\,df_j^{(e)} - x_j\,df_i^{(e)}$$

$$\iff \frac{d}{dt}\int_V (x_i\dot{x}_j - x_j\dot{x}_i)\,dm = \int_V x_i\,df_j^{(e)} - x_j\,df_i^{(e)}$$



$$\alpha_i k(\alpha_j - \beta_j) - \alpha_j k(\alpha_i - \beta_i)$$
$$+ \beta_i k(\beta_j - \alpha_j) - \beta_j k(\beta_i - \alpha_i)$$
$$= 0$$
$$\therefore \int_V x_i\,df_j^{(i)} - x_j\,df_i^{(i)} = 0$$

$$x_i \dot{x}_j - x_j \dot{x}_i = (x_{Gi} + \tilde{x}_i)(\dot{x}_{Gj} + \dot{\tilde{x}}_j) - (x_{Gj} + \tilde{x}_j)(\dot{x}_{Gi} + \dot{\tilde{x}}_i)$$

$$= (x_{Gi}\dot{x}_{Gj} - x_{Gj}\dot{x}_{Gi}) + (\tilde{x}_i \dot{\tilde{x}}_j - \tilde{x}_j \dot{\tilde{x}}_i) + \underbrace{\dot{x}_{Gj}\tilde{x}_i + x_{Gi}\dot{\tilde{x}}_j - x_{Gj}\dot{\tilde{x}}_i - \dot{x}_{Gi}\tilde{x}_j}_{\int_V \cdots \, dm = 0}$$

$$\therefore \int_V (x_i \dot{x}_j - x_j \dot{x}_i)\, dm = (x_{Gi}\dot{x}_{Gj} - x_{Gj}\dot{x}_{Gi}) \int_V dm + \int_V (\tilde{x}_i \dot{\tilde{x}}_j - \tilde{x}_j \dot{\tilde{x}}_i)\, dm$$

$$\int_V (x_i\, df_j^{(e)} - x_j\, df_i^{(e)}) = x_{Gi}\int_V df_j^{(e)} - x_{Gj}\int_V df_i^{(e)} + \int_V \tilde{x}_i\, df_j^{(e)} - \tilde{x}_j\, df_i^{(e)}$$

ここで、

$$\frac{d}{dt}(x_{Gi}\dot{x}_{Gj} - x_{Gj}\dot{x}_{Gi}) \int_V dm = (x_{Gi}\ddot{x}_{Gj} - x_{Gj}\ddot{x}_{Gi}) \int_V dm$$

$$= x_{Gi}\int_V df_j^{(e)} - x_{Gj}\int_V df_i^{(e)}$$

$$\therefore \boxed{\frac{d}{dt}\int_V (\tilde{x}_i \dot{\tilde{x}}_j - \tilde{x}_j \dot{\tilde{x}}_i)\, dm = \int_V \tilde{x}_i\, df_j^{(e)} - \tilde{x}_j\, df_i^{(e)}}$$

$\dot{\tilde{x}} = \dot{R}\tilde{x}'$

$$\dot{\tilde{x}}_i = \dot{R}_i \tilde{x}' \quad \left(R = \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix} \right)$$

$$= \dot{R}_i\, {}^t\!R R \tilde{x}' \quad \left(\dot{R}\,{}^t\!R = \begin{bmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nn} \end{bmatrix} \right)$$

$$= [w_{i1} \cdots w_{in}]\tilde{x}$$

$$= \sum_{k=1}^{n} w_{ik}\tilde{x}_k$$

($i \neq j$ で考える)

$$\tilde{x}_i \dot{\tilde{x}}_j - \tilde{x}_j \dot{\tilde{x}}_i$$

$$= \tilde{x}_i \sum_{k=1}^{n} w_{jk}\tilde{x}_k - \tilde{x}_j \sum_{k=1}^{n} w_{ik}\tilde{x}_k$$

$$= \sum_{k=1}^{n} (\tilde{x}_i \tilde{x}_k w_{jk} - \tilde{x}_j \tilde{x}_k w_{ik})$$

$$= \sum_{\substack{1 \le k \le n \\ k \neq i,j}} (\tilde{x}_i \tilde{x}_k w_{jk} - \tilde{x}_j \tilde{x}_k w_{ik}) + (\tilde{x}_i^2 + \tilde{x}_j^2) w_{ji}$$

$j < i$ として.

$$= \sum_{1 \le k < j} (-\tilde{x}_k \tilde{x}_i w_{kj} + \tilde{x}_k \tilde{x}_j w_{ki})$$

$$+ \sum_{j < k < i} (\tilde{x}_k \tilde{x}_i w_{jk} + \tilde{x}_j \tilde{x}_k w_{ki})$$

$$+ \sum_{i < k \le n} (\tilde{x}_i \tilde{x}_k w_{jk} - \tilde{x}_j \tilde{x}_k w_{ik}) + (\tilde{x}_i^2 + \tilde{x}_j^2) w_{ji}$$

$$= \left[\cdots\cdots\ (\tilde{x}_i^2 + \tilde{x}_j^2)\ \cdots\cdots \right] \begin{bmatrix} w_{12} \\ \vdots \\ w_{ji} \\ \vdots \\ w_{n-1\,n} \end{bmatrix} \qquad \therefore \frac{d}{dt}\left(\int_V [\cdots (\tilde{x}_i^2 + \tilde{x}_j^2) \cdots]\, dm \begin{bmatrix} w_{12} \\ \vdots \\ w_{ji} \\ \vdots \\ w_{n-1\,n} \end{bmatrix} \right) = \int_V \tilde{x}_i\, df_j^{(e)} - \tilde{x}_j\, df_i^{(e)}$$

行列にまとめる。

$$\boxed{\frac{d}{dt}(\underset{(nC_2 \times nC_2)}{I}\ \underset{(nC_2 \times 1)}{W}) = \underset{(nC_2 \times 1)}{N}}$$

$$W = \begin{array}{c} 12 \\ \vdots \\ ji \\ \vdots \\ n\text{-}1\,n \end{array} \begin{bmatrix} w_{12} \\ \vdots \\ w_{ji} \\ \vdots \\ w_{n-1\,n} \end{bmatrix} \Leftrightarrow N = \begin{bmatrix} \vdots \\ \int_V \tilde{x}_i\, df_j^{(e)} - \tilde{x}_j\, df_i^{(e)} \\ \vdots \end{bmatrix}$$

参考

$$\begin{cases} S(j,i) := \tilde{x}_i^2 + \tilde{x}_j^2 \\ t(j,i) := \tilde{x}_i \tilde{x}_j \end{cases}$$

$n=2$ $$I = \int_V \left[S(1,2) \right] dm$$

$n=3$ $$I = \int_V \begin{bmatrix} S(1,2) & t(2,3) & -t(1,3) \\ t(2,3) & S(1,3) & t(1,2) \\ -t(1,3) & t(1,2) & S(2,3) \end{bmatrix} dm$$

$n=4$ $$I = \int_V \begin{bmatrix} S(1,2) & t(2,3) & t(2,4) & -t(1,3) & -t(1,4) & 0 \\ t(2,3) & S(1,3) & t(3,4) & t(1,2) & 0 & -t(1,4) \\ t(2,4) & t(3,4) & S(1,4) & 0 & t(1,2) & t(1,3) \\ -t(1,3) & t(1,2) & 0 & S(2,3) & t(3,4) & -t(2,4) \\ -t(1,4) & 0 & t(1,2) & t(3,4) & S(2,4) & t(2,3) \\ 0 & -t(1,4) & t(1,3) & -t(2,4) & t(2,3) & S(3,4) \end{bmatrix} dm$$

$n=5$ $$I = \int_V \begin{bmatrix} S(1,2) & t(2,3) & t(2,4) & t(2,5) & -t(1,3) & -t(1,4) & -t(1,5) & 0 & 0 & 0 \\ t(2,3) & S(1,3) & t(3,4) & t(3,5) & t(1,2) & 0 & 0 & -t(1,4) & -t(1,5) & 0 \\ t(2,4) & t(3,4) & S(1,4) & t(4,5) & 0 & t(1,2) & 0 & t(1,3) & 0 & -t(1,5) \\ t(2,5) & t(3,5) & t(4,5) & S(1,5) & 0 & 0 & t(1,2) & 0 & t(1,3) & t(1,4) \\ -t(1,3) & t(1,2) & 0 & 0 & S(2,3) & t(3,4) & t(3,5) & -t(2,4) & -t(2,5) & 0 \\ -t(1,4) & 0 & t(1,2) & 0 & t(3,4) & S(2,4) & t(4,5) & t(2,3) & 0 & -t(2,5) \\ -t(1,5) & 0 & 0 & t(1,2) & t(3,5) & t(4,5) & S(2,5) & 0 & t(2,3) & t(2,4) \\ 0 & -t(1,4) & t(1,3) & 0 & -t(2,4) & t(2,3) & 0 & S(3,4) & t(4,5) & -t(3,5) \\ 0 & -t(1,5) & 0 & t(1,3) & -t(2,5) & 0 & t(2,3) & t(4,5) & S(3,5) & t(3,4) \\ 0 & 0 & -t(1,5) & t(1,4) & 0 & -t(2,5) & t(2,4) & -t(3,5) & t(3,4) & S(4,5) \end{bmatrix} dm$$

$$I := \int_V \begin{array}{c} \\ 12 \\ \vdots \\ ji \\ \vdots \\ n\text{-}1\,n \end{array} \begin{array}{c} \begin{array}{ccccc} 12 & \cdots & uv & \cdots & n\text{-}1\,n \end{array} \\ \begin{bmatrix} & & \vdots & & \\ & & \vdots & & \\ \cdots & \cdots & I(ji,uv) & & \\ & & & & \\ & & & & \end{bmatrix} \end{array} dm$$

$(u<v$ かつ $j<i$ かつ $1 \leq u,v,j,i \leq n)$

$$I(ji,uv) := \begin{cases} S(j,i) & (u=j<v=i \text{ のとき}) \\ t(v,i) & (u=j<v<i \text{ のとき}) \\ t(i,v) & (u=j<i<v \text{ のとき}) \\ -t(j,v) & (j<u=i<v \text{ のとき}) \\ -t(u,i) & (u<v=j<i \text{ のとき}) \\ t(j,u) & (j<u<v=i \text{ のとき}) \\ t(u,j) & (u<j<v=i \text{ のとき}) \\ 0 & (\text{それ以外のとき}) \end{cases}$$

また、

$I(ji,uv) = I(uv,ji)$ となり、I は対称行列

結論

$$\boxed{\ddot{x}_G \int_V dm = \int_V df^{(e)}, \quad \frac{d}{dt}(I w) = \mathbb{N}}$$

$$\ddot{x}\,dm = d\mathbb{f} \quad \Rightarrow \quad \dot{x}\cdot\ddot{x}\,dm = \dot{x}\cdot d\mathbb{f}$$

$$\Rightarrow \quad \int_V \dot{x}\cdot\ddot{x}\,dm = \int_V \dot{x}\cdot d\mathbb{f}$$

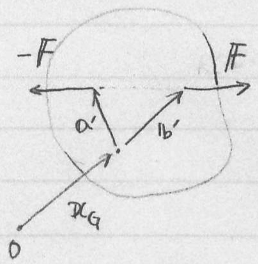
$$\Leftrightarrow \quad \int_V \frac{d}{dt}\left(\frac{1}{2}\dot{x}\cdot\dot{x}\right)dm = \int_V \dot{x}\cdot d\mathbb{f}$$

$$\Leftrightarrow \quad \frac{d}{dt}\int_V \frac{1}{2}\left(\dot{x}_G+\dot{R}\tilde{x}'\right)\cdot\left(\dot{x}_G+\dot{R}\tilde{x}'\right)dm = \int_V\left(\dot{x}_G+\dot{R}\tilde{x}'\right)\cdot\left(d\mathbb{f}^{(i)}+d\mathbb{f}^{(e)}\right)$$

$$\Leftrightarrow \frac{d}{dt}\left\{\frac{1}{2}|\dot{x}_G|^2\int_V dm + \dot{x}_G\cdot\left(\dot{R}\underbrace{\int_V\tilde{x}'dm}_{=0}\right)+\frac{1}{2}\int_V|\dot{R}\tilde{x}'|^2dm\right\}$$

$\boxed{{}^{\forall}\mathbb{A}\in\mathbb{R}^n \quad \mathbb{A}=R\mathbb{A}'}$

$$= \dot{x}_G\cdot\underbrace{\int_V d\mathbb{f}^{(i)}}_{=0} + \dot{x}_G\cdot\int_V d\mathbb{f}^{(e)} + \underbrace{\int_V(\dot{R}\tilde{x}')\cdot d\mathbb{f}^{(i)}}_{=0} + \int_V(\dot{R}\tilde{x}')\cdot d\mathbb{f}^{(e)}$$



$$(\dot{R}\mathbb{a})\cdot(-\mathbb{F}) + (\dot{R}\mathbb{b}')\cdot\mathbb{F}$$
$$=\{\dot{R}(\mathbb{b}'-\mathbb{a}')\}\cdot\mathbb{F}$$
$$=\{\dot{R}(\mathbb{b}'-\mathbb{a}')\}\cdot k(\mathbb{b}-\mathbb{a})$$
$$=k\{\dot{R}(\mathbb{b}'-\mathbb{a}')\}\cdot\{R(\mathbb{b}'-\mathbb{a}')\}$$
$$=k\frac{d}{dt}\left[\frac{1}{2}\{R(\mathbb{b}'-\mathbb{a}')\}\cdot\{R(\mathbb{b}'-\mathbb{a}')\}\right]$$
$$=\frac{k}{2}\frac{d}{dt}|R(\mathbb{b}'-\mathbb{a}')|^2$$
$$=\frac{k}{2}\frac{d}{dt}|\mathbb{b}-\mathbb{a}|^2 = 0$$

$(\mathbb{b}-\mathbb{a})$は成分は時間変化しうるが、剛体なので、大きさは一定(時間変化しない)。

$$\therefore \int_V(\dot{R}\tilde{x}')\cdot d\mathbb{f}^{(i)} = 0$$

$$\Leftrightarrow \frac{d}{dt}\left\{\frac{1}{2}|\dot{x}_G|^2\int_V dm+\frac{1}{2}\int_V|\dot{\tilde{x}}|^2dm\right\}$$
$$= \dot{x}_G\cdot\int_V d\mathbb{f}^{(e)} + \int_V\dot{\tilde{x}}\cdot d\mathbb{f}^{(e)}$$
$$\left(= \int_V \dot{x}\cdot d\mathbb{f}^{(e)}\right)$$

ここで $\frac{d}{dt}\left\{\frac{1}{2}|\dot{x}_G|^2\int_V dm\right\} = \dot{x}_G\cdot\ddot{x}_G\int_V dm = \dot{x}_G\cdot\int_V d\mathbb{f}^{(e)}$

$$\therefore \frac{d}{dt}\left\{\frac{1}{2}\int_V|\dot{\tilde{x}}|^2dm\right\} = \int_V\dot{\tilde{x}}\cdot d\mathbb{f}^{(e)}$$

$$T := \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ji}\,(\tilde{x}_i \dot{\tilde{x}}_j - \tilde{x}_j \dot{\tilde{x}}_i)$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} w_{ji}\tilde{x}_i\right)\dot{\tilde{x}}_j + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} w_{ij}\tilde{x}_j\right)\dot{\tilde{x}}_i$$

$$= \sum_{j=1}^{n}(\dot{\tilde{x}}_j)^2 + \sum_{i=1}^{n}(\dot{\tilde{x}}_i)^2 \quad \text{(No.2 より)}$$

$$= 2\left|\dot{\tilde{x}}\right|^2$$

一方 $$T = \sum_{\substack{1\le i\le n,\ 1\le j\le n \\ i\ne j}} w_{ji}(\tilde{x}_i \dot{\tilde{x}}_j - \tilde{x}_j \dot{\tilde{x}}_i)$$

$$= 2\sum_{1\le j<i\le n} w_{ji}(\tilde{x}_i \dot{\tilde{x}}_j - \tilde{x}_j \dot{\tilde{x}}_i) \qquad (i, j \text{ の対称式より})$$

$$\therefore\ \frac{T}{2} = \left|\dot{\tilde{x}}\right|^2 = \sum_{1\le j<i\le n} w_{ji}(\tilde{x}_i \dot{\tilde{x}}_j - \tilde{x}_j \dot{\tilde{x}}_i)$$

$$\Rightarrow \int_V \left|\dot{\tilde{x}}\right|^2 dm = \sum_{1\le j<i\le n} w_{ji}\int_V (\tilde{x}_i \dot{\tilde{x}}_j - \tilde{x}_j \dot{\tilde{x}}_i)\,dm$$

$$= \sum_{1\le j<i\le n} (\boldsymbol{w}\text{ の } ji \text{ 行})(I\boldsymbol{w}\text{ の } ji \text{ 行})$$

$$= \boldsymbol{w}\cdot I\boldsymbol{w}$$

$$= {}^t\boldsymbol{w}\, I\, \boldsymbol{w}$$

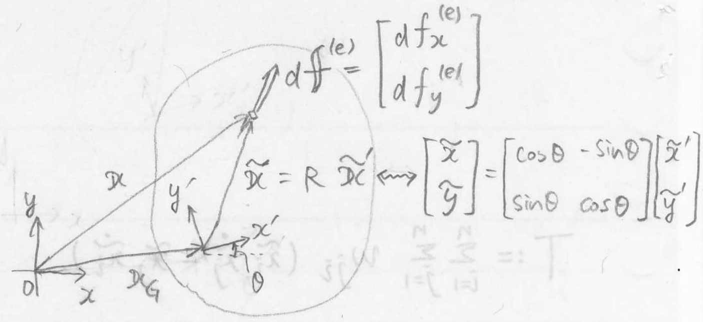
$$\therefore\ \boxed{\frac{d}{dt}\left\{\frac{1}{2}|\dot{x}_G|^2\int_V dm + \frac{1}{2}\,{}^t\boldsymbol{w}\,I\,\boldsymbol{w}\right\} = \int_V \dot{x}\cdot d\boldsymbol{f}^{(e)}}$$

外力の作用点にある微小質量の速度 vector

$$\boxed{\frac{d}{dt}\left(\frac{1}{2}\,{}^t\boldsymbol{w}\,I\,\boldsymbol{w}\right) = \int_V \dot{\tilde{x}}\cdot d\boldsymbol{f}^{(e)}}$$

外力の作用点にある微小質量の重心から見た相対速度 vector

$n=2$ のとき、さらに変形できる。



$$\frac{d}{dt}(IW) = N \quad \text{より}$$

$$\frac{d}{dt}\left(\int_V \tilde{x}^2+\tilde{y}^2\, dm\; W_{12}\right) = \int_V \tilde{y}\, df_x^{(e)} - \tilde{x}\, df_y^{(e)} \quad — ①$$

$$\dot{R}^t R = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \dot{\theta} = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} \qquad \therefore W_{12} = -\dot{\theta}$$

$$I = \int_V (\tilde{x}^2+\tilde{y}^2)\, dm = \int_V |\tilde{x}|^2\, dm$$

$$= \int_V |\tilde{x}'|^2\, dm$$ (回転変換は大きさを変えない。)

$$= \int_V (\tilde{x}'^2+\tilde{y}'^2)\, dm$$

よって、Iは時間変化しない。

これらを用いて、①を変形

$$\boxed{\int_V (\tilde{x}'^2+\tilde{y}'^2)\, dm\; \ddot{\theta} = \int_V \tilde{x}\, df_y^{(e)} - \tilde{y}\, df_x^{(e)}}$$