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Content :

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NUMBERS COMPUTED

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Numbers Computed

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Image of long bit computation (Fixed decimal, 4 place denary)

Example of multiplication : 1.23456789 × 3.45678901

1.	2 3 4 5	6 7 8 9		
3.	4 5 6 7	8 9 0 1		
3				
	7 0 3 5			
	2	0 3 6 7		
	4 5 6 7			
	1 0 7 0	9 6 1 5		
		3 1 0 0	5 3 6 3	
		8 9 0 1		
		2 0 8 7	2 8 4 5	
			6 0 4 2	8 8 8 9
4.	2 6 7 6	4 0 7 1	4 2 5 0	8 8 8 9

Image of subtraction

Example :

3. 8 9 1 2 3 7 6 2 − 1. 2 7 3 5 4 6 2 2

= 3. 8 9 1 2 3 7 6 2 + (- 2 + 0. 7 2 6 4 5 3 7 8)

= 4. 6 1 7 6 9 1 4 0 - 2

= 2. 6 1 7 6 9 1 4 0

Division is replaced by multiplication:

$$\frac{1}{1-x} = \prod_{k=0}^{\infty}\left(1+x^{2^k}\right) \text{ , } (|x|<1)$$

$$\frac{1}{1+x} = (1-x)\prod_{k=1}^{\infty}\left(1+x^{2^k}\right) \text{ , } (|x|<1)$$

1) Is this number $\exp\left\{\frac{\pi\sqrt{163}}{3}\right\}$ an integer?

The said value is approximately 640320.000 by a personal calculator. In this case computation is based on approximate expressions with finite significant figures, say 14 places.
To avoid handling sufficiently large numbers compared with unity, let us seek 1/500000 of the said value.
For this purpose, it is necessary to compute, with sufficient long decimal bits, the following

$$\frac{\pi\sqrt{163}}{3} - \ln 500000 \tag{1}$$

Note that

$$\frac{\pi\sqrt{163}}{3} = \frac{13\pi}{3}\sqrt{1-\frac{6}{169}} \tag{2}$$

$$\begin{aligned} \ln 500000 &= 2\ln 1000 - \ln 2 \\ &= 2\ln\frac{1000}{1024} + 19\ln 2 \\ &= 2\ln\frac{125}{128} + 19\ln 2 \end{aligned} \tag{3}$$

π itself can be easily computed using the expansion formulus of the arctangent up to the 10000 significant figures. Computation for $\sqrt{}$ can be based on the exponential function and the logarithmic function, or else the following binary expansion may be used.

$$(1+x)^m = 1 + \sum_{r=1}^{\infty}\left\{\frac{1}{r!}\prod_{k=0}^{r-1}(m-k)\right\}x^r\ ,\ \ (|x|<1) \tag{4}$$

For the logarithmic function the following holds:

$$\frac{1}{2}\ln\frac{1+x}{1-x} = \sum_{k=0}^{\infty}\frac{1}{2k+1}x^{2k+1}\ ,\ (|x|<1) \tag{5}$$

For the exponential function the following holds:

$$\exp x = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} x^n \tag{6}$$

P.S. π is computed through

$$\pi = 16 \tan^{-1}\left(\frac{1}{4}\right) - 8 \tan^{-1}\left(\frac{1}{38}\right) - 4 \tan^{-1}\left(\frac{1}{7}\right) \tag{7}$$

$$\tan^{-1} x = \sum_{n=1}^{\infty} \frac{1}{4n-3} x^{4n-3} - \sum_{n=1}^{\infty} \frac{1}{4n-1} x^{4n-1} \quad (|x| < 1) \tag{8}$$

The value of Eq.(1) becomes

$$0.247359952973177790425977934320038007 1024 \cdots \tag{9}$$

The value of the expression $\exp\left(\frac{\pi\sqrt{163}}{3} - \ln 500000\right)$ is

$$1.28064000000000120972747009803 2 \cdots \tag{10}$$

Thus the original expression $\exp\left(\frac{\pi\sqrt{163}}{3}\right)$ is

$$640320.000000000604863 \cdots \tag{11}$$

That is, the said value is different from an integer. By the way $\exp\left(\pi\sqrt{163}\right) = \left\{\exp\left(\pi\sqrt{163}/3\right)\right\}^3 \approx 262 \cdots 743.999999999999$.

2) Is this value $\sin\left(\frac{\pi}{\tan^{-1} 2}\right)$ a (fine) rational number?

On a personal calculator, it looks like

$$0.299377455 \tag{12}$$

$$0.200300400 \cdots = 10^2 \sum_{n=2}^{\infty} n 10^{-3(n-1)}$$

$$= 10^2 \left[\frac{2x}{1-x} + \frac{x^2}{(1-x)^2}\right]_{x\,=\,10^{-3}} = \frac{0.2}{0.999} + \frac{0.0001}{0.999^2} \tag{13}$$

$$0.09900770055\cdots = 11\sum_{n=1}^{\infty}(11-2n)\,10^{-3n}$$

$$= \left[\frac{121x}{1-x} - \frac{22\times 10^{-3}}{(1-x)^2}\right]_{x\,=\,10^{-3}} = \frac{0.121}{0.999} - \frac{0.022}{0.999^2} \tag{14}$$

If it is assumed to be a regulative number, say ϕ, then

$$\phi = \frac{0.221}{0.999} + \frac{0.078}{0.999^2} \tag{15}$$

Since $999^2/3 = 37^2\times 3^5$, the length of recurrence of ϕ in the decimal expression is a divisor of $37\times 36\times 3^4\times 2 = 215784$. Instead we shall check whether

$$\tan\left(\frac{\pi}{\pi-\sin^{-1}\phi}\right) \tag{16}$$

is equal to 2 or not. Since actually $\dfrac{\pi}{\pi-\sin^{-1}\phi} > 1$, use is made of the following identity:

$$\tan\frac{\pi}{\pi-\sin^{-1}\phi} = \frac{1+\tan\left(\frac{\pi}{\pi-\sin^{-1}\phi}-\frac{\pi}{4}\right)}{1-\tan\left(\frac{\pi}{\pi-\sin^{-1}\phi}-\frac{\pi}{4}\right)} \tag{17}$$

$$\tan\left(\frac{\pi}{\pi-\sin^{-1}\phi}-\pi/4\right) \equiv \frac{\sin\left(\frac{\pi}{\pi-\sin^{-1}\phi}-\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{\pi-\sin^{-1}\phi}-\frac{\pi}{4}\right)} \tag{18}$$

In adition to this

$$\sin^{-1}x = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n+1)(2n)!!}x^{2n+1} \tag{19}$$

Finally the value of Eq.(16) becomes

$$2.000000018157 \tag{20}$$

That is, the said value is diferent from a fine number ϕ.

3) How to compute wisely γ(Euler constant) $= \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)$ It is still not yet known whether γ

is a transcendental number.

After computing up to $n = 10^m$, the constant will be determined roughly within m digits. Let us try to use an identity

$$\gamma = \int_0^\infty \frac{1}{x} \left\{ \frac{2}{\pi} \cot^{-1} x - \exp(-x) \right\} dx \tag{21}$$

which is identical to

$$\gamma = \int_0^1 \frac{1}{x} \{1 - \exp(-x) - \exp(-1/x)\} dx \tag{22}$$

A part of the integral

$$\int_0^1 \frac{1}{x} \{1 - \exp(-x)\} dx = 1 - \frac{1}{2 \cdot 2!} + \frac{1}{3 \cdot 3!} - \cdots \tag{23}$$

is easily numerically evaluated to give

$$0.796599599297053134283675865542524080073206629346 83 \tag{24}$$

The remaining part $\int_0^1 \exp(-1/x)/x dx$ is evaluated using numerical integration based on trapezoidal formula. If the interval $[0, 1]$ is equally divided into 100 parts, then the integral is roughly

$$0.219383934396 \tag{25}$$

so that

$$\gamma = 0.577215664901 \tag{26}$$

4) Numerical computation relating to prime numbers

$$\begin{aligned} \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{4n}} &= \ln \zeta(4) \\ &= \ln \left(\frac{\pi^4}{90} \right) \end{aligned} \tag{27}$$

where summation on p stands for summation over all primes. The value of the left hand side up to the prime number 27449 is

$$0.079109873067334104248614109298633367917140867411 78 \quad (28)$$

On the other hand, the right hand side of Eq.(27) is approximately (by computer)

$$0.07910987306733385 \quad (29)$$

and due to computation of long bits

$$0.07910987306733562976522747688281922969309664738697 \quad (30)$$

5) Is this value $\tan\left(\dfrac{\pi}{2\tan^{-1} 2}\right)$ a (fine) rational number? < Case 2 >

The approximate value is

$$\underline{6.5}\ 2\ \underline{73}\ 2\ \underline{77}\ 2 \quad (31)$$

and its supposed value ϕ is given by

$$\begin{aligned}\phi &= \frac{8.12}{0.999} - \frac{1.6}{0.9995} \\ &= \frac{8120}{999} - \frac{3200}{1999}\end{aligned} \quad (32)$$

In this case we will check whether the following expression

$$\tan\left(\frac{\pi}{2\tan^{-1}\phi}\right) \quad (33)$$

is equal to 2 or not. Actually

$$\tan\left(\frac{\pi}{2\tan^{-1}\phi}\right) = \frac{\cos\psi + \sin\psi}{\cos\psi - \sin\psi} \quad (34)$$

where

$$\psi \equiv \frac{1}{1-\frac{2}{\pi}\tan^{-1}\left(\frac{1}{\phi}\right)} - \frac{\pi}{4} \tag{35}$$

Then Eq.(33) is evaluated to be

$$1.999999999661058907792811837565650157343733181177 21 \tag{36}$$

which is different from 2.

6) Fine number?

$$\tan\left\{\frac{\pi}{2}\Big/\exp\left(\frac{1}{\pi\sqrt{118}}\right)\right\} \tag{37}$$

This value can be computed as follows:

$$(\text{the said expression}) = \frac{\cos\left\{\frac{\pi}{2}\left(1-\mathrm{e}^{-\phi}\right)\right\}}{\sin\left\{\frac{\pi}{2}\left(1-\mathrm{e}^{-\phi}\right)\right\}} \tag{38}$$

$$\begin{aligned} \phi &\equiv \frac{1}{\pi\sqrt{118}} \\ &= \frac{1}{11\pi\sqrt{118/121}} \end{aligned} \tag{39}$$

$$\sqrt{\frac{118}{121}} = \exp\left(\frac{1}{2}\ln\frac{239-3}{239+3}\right) \tag{40}$$

The result is

$$22.03030303875326265709038596862305698 \tag{41}$$